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An Optimal Interpolation Formula

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1. INTRODUCTION

Let F be a given function class and let the functionals $L(f), L_1(f),..., L_N(f)$ be defined on F. In recent years the problem of constructing methods for the approximate evaluation of the functional $L(f)$, $f \in F$ which use only the information

$$T(f) = \{L_1(f),..., L_N(f)\}$$

and possess certain good extremal properties, became a subject of investigation for many authors.

We shall search for a method of approximation of $L(f)$ with minimal error in the class F. Precisely, let $\sigma(L, T)$ denote the set of all methods S which are generated by a real function $S(t_1, ..., t_N)$ of N variables as

$$S : L(f) \approx S(L_1(f),..., L_N(f)) = S(f)$$

for every $f \in F$. A method σ^* for which

$$\sup_{f \in F} | L(f) - S^*(f) | = \inf_{S \in \sigma(L,T)} \sup_{f \in F} | L(f) - S(f) |$$

is said to be best for the class F with respect to the information T.

In the case when $L(f) = f(x)$, x is fixed in $[a, b]$, and $T(f)$ consists of values of the function f and its derivatives at n discrete points $x_1, ..., x_n$, the error

$$R := \sup_{f \in F} | f(x) - S^*(f) |$$

of the best method depends on x and $\{x_k\}_1^n$: $R = R(x; x_1, ..., x_n)$ (occasionally shortened to $R(x)$). Let $\| R \| = \sup_{t \in (a,b)} | R(t; x_1, ..., x_n)|$. The nodes for which $\| R \|$ attains its minimal value are called optimal. Let $S(f)(x)$ be the approximate value of $f(x)$ calculated by a best method with optimal nodes.

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Clearly $S(f)(x)$ is a function of x. The method

$$f(x) \approx S(f)(x) \qquad \text{for } x \in [a, b]$$

is an optimal method of approximation of $f(x)$ for the class F.

The extremal interpolation problem stated in this way was solved entirely in [1] for F, the set of all real functions in $[-1, 1]$ which are analytic and bounded by 1 in the unit circle,

$$T(f) = \{f^{(k)}(x_i), i = 1, 2, \ldots, n; k = 0, 1, \ldots, m\},$$
$$-1 < x_1 < \cdots < x_n < 1,$$

and in [2] for

$$F = W_q^{(r)}(M; [a, b]) := \{f \in C^{(r-1)}[a, b]: f^{(r-1)} \text{ abs. const.}, \ \|f^{(r)}\|_q \leq M\},$$
$$r = 1, 2, \ldots, \quad 1 \leq q \leq \infty,$$
$$T(f) = \{f^{(k)}(x_i), i = 1, 2, \ldots, n; k = 0, 1, \ldots, r-1\},$$
$$a \leq x_1 < \cdots < x_n \leq b.$$

Perhaps the most interesting case apears when

$$T(f) = \{f(x_0), f(x_1), \ldots, f(x_n)\}$$

and $F = W_q^{(r)}(M; [a, b])$, $r = 1, 2, \ldots$. It follows from the results obtained in [2] that for $r = 1$ and nodes $\{x_k\}_0^n$ fixed in $[a, b]$ the best method of approximation of $f(x)$ is

$$f(x) \approx f(x_k) \qquad \text{for} \quad x: |x - x_k| = \inf_i |x - x_i|.$$

The optimal nodes are also determined.

The purpose of this paper is to construct an optimal method of approximation of $f(x)$ when $r = 2$, $q = \infty$. We show that the optimal method is an interpolation of $f(x)$ at the nodes by a polygonal line.

2. Preliminaries

Our study in the sequel is based on the following lemma of Smoljak [3] (see also [4]).

LEMMA 1. *Let H be a linear metric space and F is convex centrally symmetrical set in H with a center of symmetry O. Suppose the functionals $L(f)$, $\{L_k(f)\}_1^N$ are linear and defined on H. If* $\sup\{L(f) : f \in F_0\} < \infty$ *where*

$$F_0 := \{f \in F : L_k(f) = 0, k = 1, 2, \ldots, N\}.$$

then there exist numbers $C_1, C_2, \dots, C_N$ *such that*

$$R = \sup_{f \in F} \left| L(f) - \sum_{k=1}^{N} C_k L_k(f) \right|$$

is equal to the error of the best method, i.e., there is a linear best method of approximation of $L(f)$.

The proof of this lemma contains a useful result which we shall formulate as a separate proposition.

COROLLARY 1. $R = \sup\{L(f) : f \in F_0\}$.

As usual $W_\infty^{(r)}[a, b]$ will denote the Sobolev space

$$\{f \in C^{(r-1)}[a, b] : f^{(r-1)} \text{ abs. cont.}, f^{(r)} \in L_\infty[a, b]\}.$$

We shall make use of the following result of Karlin [5].

THEOREM A. *Let the points* $\{x_k\}_1^N$ *satisfy*

$$a \leq x_1 < \cdots < x_N \leq b$$

and $\{y_i\}_1^N$ *be a given sequence of real numbers. The extremal problem*

$$\min\{\|f^{(r)}\|_\infty : f \in \Omega_0\},$$

$$\Omega_0 := \{f \in W_\infty^{(r)}[a, b] : f(x_i) = y_i,\ i = 1, 2, \dots, N\},$$

has a solution of the form

$$\sum_{i=0}^{r-1} a_i t^i + c \left[t^r + 2 \sum_{i=1}^{k-1} (-1)^i (t - \xi_i)_+^r \right] \qquad (1)$$

for some real constants $a_0, \dots, a_{r-1}$ *and* c, *and for* $a < \xi_1 < \cdots < \xi_{k-1} < b$ *with* $k \leq N - r$.

Karlin has proved this theorem in a more general form, which allows for the interpolation of consecutive derivatives as well. Another proof can be found in [6]. Note that function (1) is called perfect spline.

3. Main Results

First we shall prove

LEMMA 2. *Suppose $x \in [a, b]$ and α, β be such that $a \leq \alpha < \beta \leq b$. Then the function $\varphi(t; \alpha, \beta) = M(t - \alpha)(\beta - t)/2$ is a solution of the extremal problem*

$$\sup\{|f(x)| : f \in F(\alpha, \beta)\},$$

$$F(\alpha, \beta) := \{f \in W_\infty^{(2)}(M; [a, b]) : f(\alpha) = f(\beta) = 0\}.$$

Proof. Let $f[x, \alpha, \beta]$ denote the divided difference of the function $f(t)$ based on the points x, α, β. By Peano's theorem we get the integral representation

$$f[x, \alpha, \beta] = \int_a^b \tau(t; x, \alpha, \beta) f''(t)\, dt \tag{2}$$

where

$$\tau(t) = \tau(t; x, \alpha, \beta) = \frac{(x - t)_+}{(x - \alpha)(x - \beta)} - \frac{(\alpha - t)_+}{2h(\alpha - x)} + \frac{(\beta - t)_+}{2h(\beta - x)},$$

$2h = \beta - \alpha$ and the truncated function u_+ is defined by $u_+ = \max(u, 0)$. Let us assume now that $f \in F(\alpha, \beta)$. Then, from (2)

$$f(x) = (x - \alpha)(x - \beta) \int_a^b \tau(t) f''(t)\, dt$$

and consequently

$$|f(x)| \leq M\, |(x - \alpha)(x - \beta)| \int_a^b |\tau(t)|\, dt. \tag{3}$$

Expression (2) for $f[x, \alpha, \beta]$ is a special case of the formula obtained by Tschakaloff [7, Theorem 2] for the divided difference of the function f based on the points $t_1, t_2, \ldots, t_m$ with multiplicities $\nu_1, \nu_2, \ldots, \nu_m$, respectively. It follows by the cited theorem that $\tau(t) \geq 0$ for all t and $\int_a^b \tau(t)\, dt = \frac{1}{2}$. That and (3) give

$$\sup_{f \in F(\alpha, \beta)} |f(x)| \leq |\varphi(x; \alpha, \beta)|. \tag{4}$$

Now we observe that $\varphi(x; \alpha, \beta) \in F(\alpha, \beta)$; the upper bound in (4) is actually attained. The proof is complete.

THEOREM 1. *Let the nodes $\{\xi_k\}_0^n$ be defined by*

$$\xi_k = a + (2^{1/2} - 1)h + 2kh, \qquad (k = 0, 1, \ldots, n),$$
$$h = (b - a)/(2(n + 2^{1/2} - 1)).$$

The method

$$f(x) \approx l_k(x) \qquad \text{for } x \in \mathcal{T}_k, \ (k = 1, 2,\dots, n)$$

where

$$\mathcal{T}_1 = [a, \xi_1], \quad \mathcal{T}_n = [\xi_{n-1}, b], \quad \mathcal{T}_k = [\xi_{k-1}, \xi_k], \quad (k = 2, 3,\dots, n-1),$$

$$l_k(x) = f(\xi_{k-1})((x - \xi_k)/(\xi_{k-1} - \xi_k)) + f(\xi_k)((x - \xi_{k-1})/(\xi_k - \xi_{k-1})))$$

is an optimal method of approximation of $f(x)$ from the class $W_\infty^{(2)}(M; [a, b])$ with respect to the information of the type

$$\{f(x_0), f(x_1),\dots, f(x_n)\}, \qquad a \le x_0 < \cdots < x_n \le b.$$

The error $R^*(x)$ of the optimal method has the value

$$R^*(x) = |M(x - \xi_{i-1})(\xi_i - x)/2| \qquad \text{for } x \in \mathcal{T}_i, \ (i = 1, 2,\dots, n)$$

and

$$\|R^*\| = M((b - a)/(n + 2^{1/2} - 1))^2/8.$$

Proof. Let $R_0(x) = R_0(x; \xi_0,\dots, \xi_n)$ denote the error of the best method of approximation of $f(x)$ based on the nodes $\{\xi_k\}_0^n$. We shall show that

$$\|R_0\| = M((b - a)/(n + (2^{1/2} - 1))^2/8. \tag{5}$$

To establish (5), define the function $\varphi(t)$ by

$$\varphi(t) = (-1)^j M(t - \xi_{j-1})(\xi_j - t)/2 \qquad \text{for } t \in \mathcal{T}_j \ (j = 1, 2,\dots, n).$$

It is clear that

$$\varphi(t) \in F_0(\boldsymbol{\xi}) := \{f \in W_\infty^{(2)}(M; [a, b]): f(\xi_k) = 0, \ k = 0, 1,\dots, n\}.$$

The same holds for the function $-\varphi(t)$. Now suppose that $x \in \mathcal{T}_i$. By Corollary 1 we have

$$R_0(x) = \sup_{f \in F_0(\boldsymbol{\xi})} |f(x)| \ge \max(\varphi(x), -\varphi(x)) = |\varphi(x)|.$$

On the other hand, by Lemma 2

$$R_0(x) \le \sup_{f \in F(\xi_{i-1}, \xi_i)} |f(x)| = |M(x - \xi_{i-1})(\xi_i - x)/2|.$$

These two inequalities give $R_0(x) = |\varphi(x)|$ for all $x \in [a, b]$. Further, after simple computations we obtain (5).

The second step of our proof is to show that the nodes $\{\xi_k\}_0^n$ are optimal.

In order to prove this let us assume that the nodes $\{x_k\}_0^n$, $a \leqslant x_0 < \cdots < x_n \leqslant b$ are optimal, i.e.,

$$\max_{t\in[a,b]} |R(t; x_0,..., x_n)| = \min_{\{y_k\}_0^n} \max_{t\in[a,b]} |R(t; y_0,..., y_n)|.$$

Therefore the error $R(x)$ of the best method based on the nodes $\{x_k\}_0^n$ satisfies the inequality

$$\| R \| \leqslant \| R_0 \| . \tag{6}$$

Let $\gamma \in [a, b]$ be such that $R(\gamma) = \| R \|$. By Corollary 1

$$R(\gamma) = \sup_{f\in F_0(\mathbf{x})} f(\gamma).$$

We shall prove that there exists a function $g(t) \in F_0(\mathbf{x})$ of the form

$$g(t) = a_0 + a_1 t + c\left[t^2 + 2\sum_{i=1}^{k-1} (-1)^i (t - y_i)_+^2\right] \tag{7}$$

where a_0, a_1, c are real numbers, $a < y_1 < \cdots y_{k-1} < b$, $k = n$ for which

$$g(\gamma) = R(\gamma). \tag{8}$$

Indeed, denote by σ the set of all functions $f \in F_0(\mathbf{x})$ for which $f(\gamma) = R(\gamma)$ and by σ_0 the set of the solutions of the extremal problem

$$\begin{gathered} \inf\{\| f'' \| : f \in \Omega_0\} \\ \Omega_0 := \{f \in W_\infty^{(2)}[a, b] : f(\gamma) = 1, f(x_k) = 0, k = 0, 1,..., n\}. \end{gathered} \tag{9}$$

It is easily seen that there is one-to-one correspondence between σ and σ_0. Moreover, for every $f \in \sigma_0$ the function $Mf(t)/\| f'' \|$ belongs to the set σ. But problem (9) has a perfect spline solution of the form (7). This follows from Theorem A stated in the beginning. Consequently there is a function $g(t)$ satisfying (8). The special form of $g(t)$ makes it possible to find the number of the zeros of $g'(t)$ in $[a, b]$. We know that $g(t)$ has at least $n + 1$ zeros: x_0,..., x_n. By Rolle's theorem it follows that $g'(t)$ has at least n zeros. But $g''(t)$ is a piecewise constant function with at most $n - 1$ discontinuities and consequently $g'(t)$ has n zeros at most. Hence $g'(t)$ has exact n zeros. Let us denote them by η_1,..., η_n. Put for convenience $\eta_1 - a = h_0$, $b - \eta_n = h_n$,

$$\eta_{i+1} - \eta_i = h_i \qquad (i = 1, 2,..., n - 1).$$

Assume that

$$h_0 > 2(\| R_0 \|)^{1/2}. \tag{10}$$

Since $\|g\| = g(\gamma) = R(\gamma) = \|R\|$, (6) gives

$$\|g\| \leqslant \|R_0\| . \tag{11}$$

Therefore $|g(a)| \leqslant \|R_0\|$. On the other hand $|g'(t)| \geqslant M(\eta_1 - t)$ for $a \leqslant t \leqslant \eta_1$. Then, by Taylor's formula

$$\begin{aligned} |g(\eta_1)| &\geqslant \left| |g(a)| - \int_a^{\eta_1} g'(t)\, dt \right| \\ &\geqslant \big|\, |g(a)| - 2\|R_0\| \,\big| \geqslant \|R_0\| . \end{aligned} \tag{12}$$

But this inequality is contradictory to (11). Thus

$$h_0 \leqslant 2(\|R_0\|)^{1/2}. \tag{13}$$

In the same way we find that

$$h_n \leqslant 2(\|R_0\|)^{1/2}. \tag{14}$$

Next we have

$$\sum_{i=1}^{n-1} h_i = b - a - h_0 - h_n \geqslant b - a - 4(\|R_0\|)^{1/2}.$$

Thus there exists at least one i, $1 \leqslant i \leqslant n - 1$, such that

$$h_i \geqslant (b - a - 4(\|R_0\|)^{1/2})/(n - 1). \tag{15}$$

Since

$$\left| \int_{\eta_i}^{\eta_{i+1}} g'(t)\, dt \right| \geqslant Mh_i^2/4,$$

and remembering that η_i and η_{i+1} are successive zeros of $g'(t)$, we have $|g(\eta_i)| + |g(\eta_{i+1})| \geqslant Mh_i^2/4$. It follows from this relation that one of the quantities $|g(\eta_i)|$ or $|g(\eta_{i+1})|$ must be greater than or equal to $Mh_i^2/8$. Suppose that

$$|g(\eta_i)| \geqslant Mh_i^2/8 \geqslant \frac{M(b - a - 4(\|R_0\|)^{1/2})^2}{8(n - 1)^2}. \tag{16}$$

On the other hand, from (11) $|g(\eta_i)| \leqslant \|R_0\|$. So

$$M((b - a - 4(\|R_0\|)^{1/2})/(n - 1))^2/8 \leqslant \|R_0\| .$$

An elementary calculation gives

$$M((b - a)/(n + 2^{1/2} - 1))^2/8 \leqslant \|R_0\| . \tag{17}$$

There are now two cases:

(a) At least one of the inequalities (13), (14), of (15) is strict;

(b) none of (13), (14), and (15) is strict.

In case (a) (16) and consequently (17) will be strict and that will contradict with (5). Therefore

$$h_0 = h_n = 2(\|R_0\|)^{1/2},$$
$$h_i = (b - a - 4(\|R_0\|)^{1/2})/(n - 1).$$

Then we have from (11) $\|R_0\| \geq |g(a)| - 2\|R_0\| \geq \|R_0\|$ which produces $|g(a)| = \|R_0\|$. By Taylor's formula

$$g(x) = \|R_0\| + \int_a^x g'(t)\, dt.$$

Taking into account that $g'(t)$ is a continuous piecewise linear function with $|g''(t)| = M$ and $g'(\eta_i) = 0$ $(i = 1, 2,..., n)$ we obtain

$$g(\xi_i) = 0 \qquad (i = 0, 1,..., n).$$

But $g(t)$ has at most $n + 1$ zeros and they are equal to $\{x_k\}_0^n$. Thus $\xi_i = x_i$ $(i = 0, 1,..., n)$. So we get that the nodes $\{\xi_k\}_0^n$ are optimal. The second part of our theorem is proved. It remains to construct the optimal method. i.e., the best method for the nodes $\{\xi_k\}_0^n$.

In order to do that let us carefully consider the following relation which follows by using the function $\varphi(t)$ defined in the first part of the proof.

$$\sup_{f \in F_0(\xi)} f(x) = \sup_{f \in F(\xi_i, \xi_{i+1})} f(x) \qquad \text{for} \quad x \in \mathscr{T}_{i+1} \quad (i = 0, 1,..., n - 1).$$

This, together with Corollary 1, shows that the error of the best method of approximation of $f(x)$, $x \in \mathscr{T}_{i+1}$ based on the points ξ_i and ξ_{i+1} only, is equal to the error of the optimal method. So our problem reduces to the simple one: For every x from $\mathscr{T}_{i+1}$, $i = 0, 1,..., n - 1$ we must construct the best method

$$f(x) \approx S(f)(x)$$

with respect to the information $\{f(\xi_i), f(\xi_{i+1})\}$. Because of Lemma 1 we can confine ourself to the linear methods. First we see that the best method must be precise for polynomials of degree less than or equal to 1. Indeed, suppose that the method

$$f(x) \approx A_i(x) f(\xi_i) + B_i(x) f(\xi_{i+1}), \qquad \text{for } x \in \mathscr{T}_{i+1}$$

is a best one. Making use of Taylor's formula we get

$$\begin{aligned} f(x) - (A_i(x) f(\xi_i) + B_i(x) f(\xi_{i+1})) &= (1 - A_i(x) - B_i(x)) f(a) + (x - a - A_i(x)(\xi_i - a) \\ &\quad - B_i(x)(\xi_{i+1} - a)) f'(a) + \int_a^b \theta(t) f''(t)\, dt \end{aligned}$$

where the function $\theta(t)$ is bounded. The expressions in front of $f(a)$ and $f'(a)$ must be equal to zero, for otherwise the above expression for the error can become sufficiently large for certain linear functions. But these two conditions define $A_i(x)$ and $B_i(x)$ uniquely

$$A_i(x) = (x - \xi_{i+1})/(\xi_i - \xi_{i+1}),$$
$$B_i(x) = (x - \xi_i)/(\xi_{i+1} - \xi_i).$$

The theorem is proved.

We have found that the nodes of the optimal formula are expressed in terms of the irrational number $2^{1/2}$. This makes the formula rather difficult for practical calculations. For this reason we shall consider formulas with nodes

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b. \tag{18}$$

Then the following can be proved.

THEOREM 2. *The method*

$$f(x) \approx l_i(x) \qquad \text{for } x \in [y_i, y_{i+1}]\ (i = 0, 1, \ldots, n-1),$$

where

$$y_i = a + ih \quad (i = 0, 1, \ldots, n), \qquad h = (b - a)/n,$$
$$l_i(x) = f(y_i)(x - y_{i+1})/(y_i - y_{i+1}) + f(y_{i+1})(x - y_i)/(y_{i+1} - y_i),$$

is an optimal method of approximation of $f(x)$ from the class $W_\infty^{(2)}(M; [a, b])$ with respect to the information $\{f(x_0), \ldots, f(x_n)\}$, the nodes $\{x_k\}_0^n$ satisfying (18). *Here*

$$R(x; y_0, y_1, \ldots, y_n) = M(x - y_i)(y_{i+1} - x)/2$$

for $x \in [y_i, y_{i+1}]$ and $\|R\| = M((b - a)/n)^2/8$.

The proof is simmilar to that of the previous theorem and we omit the details.

4. Applications

First, we shall consider the problem of the order of approximation of functions by polygonal lines. We need some notations. For given numbers $t_1, ..., t_m$, let $H(t_1, ..., t_m)$ denote the set of all polygons with vertices at $t_1, ..., t_m$. Suppose

$$a \leq x_0 < x_1 < \cdots < x_{n-1} < x_n \leq b.$$

Let us denote by $p(f; x)$ the unique polygon from $H(x_1, ..., x_{n-1})$ satisfying the conditions

$$p(f; x_k) = f(x_k) \qquad (k = 0, 1, ..., n).$$

Denote else by $e_n(f)$ the quantity $\| f(x) - p(f; x) \|$. As an application of Theorem 1 we shall prove

Corollary 1.

$$\inf_{\{x_k\}_0^n} \sup_{f \in W_\infty^{(2)}(M; [a,b])} e_n(f) = M((b - a)/(n + 2^{1/2} - 1)^2/8.$$

Proof. It is seen that the method $f(x) \approx p(f; x)$, $x \in [a, b]$ uses the information $\{f(x_k)\}_0^n$ only. But the method defined in Theorem 1 is best for the class $W_\infty^{(2)}(M; [a, b])$ among all methods of this type. This proves our statement.

Subbotin and Černyh have studied in [8, Theorem 4] an analogous problem under the constraints $x_0 = a$ and $x_n = b$. In the same fashion as we proved Corollary 1, one can derive from Theorem 2 the following

Corollary 2. *If $x_0 = a$ and $x_n = b$ then*

$$\inf_{\{x_k\}_1^{n-1}} \sup_{f \in W_\infty^{(2)}(M; [a,b])} e_n(f) = M((b - a)/n)^2/8.$$

We shall now give a lower bound for the best polynomial approximation of functions from the class $W_\infty^{(2)}(M; [a, b])$. The following holds.

Corollary 3.

$$\sup_{f \in W_\infty^{(2)}(M; [a,b])} \inf_{p \in \pi_n} \| f - p \| \geq M((b - a)/(n + 2^{1/2} + 1))^2/8.$$

Proof. Let the function $\varphi(t)$ be defined as in the proof of Theorem 1. By the Chebyshev alternation theorem we have

$$\inf_{p \in \pi_n} \| \varphi - p \| = \| \varphi \|.$$

The norm of φ is calculated in the previous section. Adding sup on the left-hand side we get the result.

Remark. The general problem (for $r = 1, 2, \ldots$) of optimal interpolation was solved in [9] by Micchelli *et al.*, after we had informed Professor Micchelli about the main result of our work.

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